Unbiasedness of the Sample Mean Under Finite Expectation

Basis

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Abstract

This note establishes, under a minimal and explicitly stated assumption, that the sample mean is an unbiased estimator of the common population mean. For real-valued random variables $X_1, \dots, X_n$ with finite expectation and a common mean $\mu = \mathbb{E}[X_i]$, we prove that $\mathbb{E}[\bar{X}_n] = \mu$ by a direct application of integrability and linearity of expectation; in particular, independence is not required for unbiasedness. We also record a concrete failure mode: for heavy-tailed models such as the standard Cauchy distribution, the mean (and hence unbiasedness as an identity of finite expectations) is not well-defined.

1 Introduction

The sample mean is one of the most fundamental estimators in probability and statistics. In many applications it serves as a default point estimator for an unknown population mean, and it is often introduced alongside stronger asymptotic statements such as laws of large numbers or central limit theorems. Before such fluctuation and convergence results are invoked, however, it is useful to isolate the basic property that makes the sample mean a natural estimator: under mild conditions, its expectation agrees exactly with the target mean.

This paper gives a short, rigorous proof that the sample mean

$$\bar{X}_n := \frac{1}{n} \sum_{i=1}^{n} X_i$$

is an unbiased estimator of the common mean μ, provided the underlying random variables are integrable and share the same expectation. The key assumption is finiteness of the mean (e.g., $\mathbb{E}[|X_1|] < \infty$), which ensures that all quantities appearing in the statement are well-defined as finite real numbers and that linearity of expectation applies to the finite sum. A further point of emphasis is that independence is not needed for unbiasedness; it is relevant for distributional properties and convergence behavior, but not for the identity $\mathbb{E}[\bar{X}_n] = \mu$.

We also clarify, by a concrete example, how unbiasedness can fail in a precise sense: if the population mean does not exist as a finite real number, then there is no finite parameter μ for which the unbiasedness equation can even be posed.

Organization. Section 1 states the estimator and isolates the key assumption needed to speak meaningfully about unbiasedness. Section 2 proves unbiasedness of the sample mean by verifying integrability and applying linearity of expectation. Section 3 presents a standard Cauchy example, where the mean fails to exist and the notion of unbiasedness breaks down.

2 Statement and Key Assumption

Let $X_1, \dots, X_n$ be real-valued random variables defined on a common probability space. For unbiasedness of the sample mean, the essential assumptions are:

(i) the variables have a common finite mean, in the sense that they are identically distributed (or, more generally, satisfy $\mathbb{E}[X_i] = \mu$ for all i for some real number μ), and

(ii) the expectation exists as a finite real number, for example by assuming integrability

$$\mathbb{E}[|X_1|] < \infty,$$

which implies $\mathbb{E}[X_1] = \mu \in \mathbb{R}$ is well-defined.

Define the sample mean

$$\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Independence is *not* needed to compute $\mathbb{E}[\bar{X}_n]$; it becomes relevant for fluctuation and convergence results (e.g., laws of large numbers and central limit theorems). A common strengthening is to assume the X_i are i.i.d., which in particular implies identical distribution and thus a common mean when $\mathbb{E}[|X_1|] < \infty$.

3 Unbiasedness of the Sample Mean (Proof)

Theorem 3.1. *Assume $X_1, \dots, X_n$ are identically distributed and satisfy $\mathbb{E}[|X_1|] < \infty$. Let $\mu := \mathbb{E}[X_1] \in \mathbb{R}$ and $\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i$. Then $\bar{X}_n$ is an unbiased estimator of μ, i.e.*

$$\mathbb{E}[\bar{X}_n] = \mu.$$

Proof. Since X_i are identically distributed and $\mathbb{E}[|X_1|] < \infty$, we have $\mathbb{E}[|X_i|] = \mathbb{E}[|X_1|] < \infty$ for each i, so each X_i is integrable and $\mathbb{E}[X_i]$ is a finite real number. Moreover, the sum $\sum_{i=1}^{n} X_i$ is integrable because the triangle inequality gives

$$\mathbb{E}\Big[\Big|\sum_{i=1}^{n} X_i\Big|\Big] \le \sum_{i=1}^{n} \mathbb{E}[|X_i|] = n\,\mathbb{E}[|X_1|] < \infty.$$

Therefore linearity of expectation applies to $\sum_{i=1}^{n} X_i$, and also $\mathbb{E}[cY] = c\,\mathbb{E}[Y]$ applies for the scalar $c = 1/n$. Hence

$$\mathbb{E}[\bar{X}_n] = \mathbb{E}\Big[\frac{1}{n}\sum_{i=1}^{n} X_i\Big] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i].$$

By identical distribution, $\mathbb{E}[X_i] = \mathbb{E}[X_1] = \mu$ for every i, so

$$\mathbb{E}[\bar{X}_n] = \frac{1}{n}\sum_{i=1}^{n} \mu = \mu.$$

This calculation does not use independence; it holds under arbitrary dependence as long as the marginal means are finite and equal. $\square$

4 Concrete Failure Case

The conclusion of Theorem 3.1 is an identity of expectations, so it presupposes that the target mean μ exists as a finite real number. When $\mathbb{E}[X_1]$ is undefined (or infinite), there is no finite parameter μ for which "$\mathbb{E}[\bar{X}_n] = \mu$" can be asserted.

Example (standard Cauchy: mean does not exist). Let $X_1, \dots, X_n$ be i.i.d. standard Cauchy with density $f(x) = \frac{1}{\pi(1+x^2)}$. The mean does not exist because the positive and negative parts have infinite expectation. Writing $X_1^+ := \max\{X_1, 0\}$ and $X_1^- := \max\{-X_1, 0\}$, we have

$$\mathbb{E}[X_1^+] = \int_0^\infty x\,\frac{1}{\pi(1+x^2)}\,dx = \frac{1}{\pi}\int_0^\infty \frac{x}{1+x^2}\,dx = \frac{1}{2\pi}\int_0^\infty \frac{1}{1+u}\,du = \infty,$$

where we used the substitution $u = x^2$ (so $du = 2x\,dx$). By symmetry, $\mathbb{E}[X_1^-] = \infty$ as well. Consequently, $\mathbb{E}[X_1]$ is not defined as a finite real number (indeed, it is undefined in the Lebesgue sense since both parts are infinite), and $\bar{X}_n$ also fails to be integrable; in particular, $\mathbb{E}[\bar{X}_n]$ is not a finite number.

This illustrates that "unbiasedness" is meaningful only under integrability (e.g., $\mathbb{E}[|X_1|] < \infty$). In later developments, one typically adds further assumptions (often independence plus moment or tail conditions) to obtain convergence guarantees for $\bar{X}_n$ beyond the identity $\mathbb{E}[\bar{X}_n] = \mu$.

5 Conclusion and Outlook

Under the sole substantive requirement that the random variables have a common finite mean—for instance, $\mathbb{E}[|X_1|] < \infty$ together with identical distribution (or, more generally, $\mathbb{E}[X_i] = \mu$ for all i)—we proved that the sample mean satisfies the unbiasedness identity $\mathbb{E}[\bar{X}_n] = \mu$. The proof is purely measure-theoretic: integrability guarantees that the finite sum is integrable, and linearity of expectation then yields the result. Notably, no independence assumption is needed for this expectation calculation.

We also exhibited a concrete failure case: for i.i.d. standard Cauchy observations, the mean is undefined because both the positive and negative parts have infinite expectation. In such settings, the equation "$\mathbb{E}[\bar{X}_n] = \mu$" cannot be asserted for any finite μ, since $\mathbb{E}[\bar{X}_n]$ is not a finite real number.

A natural next step, beyond unbiasedness, is to study performance guarantees that do require additional structure, such as independence and suitable moment or tail assumptions, to obtain consistency, concentration bounds, or asymptotic normality for $\bar{X}_n$.